



TITLE:

余次元2ではめ込まれた多様体の多重点と同境類について (局所的及び大域的特異点論の研究)

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余次元 2 ではめ込まれた多様体の多重点と同境界類について

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1. INTRODUCTION AND KNOWN RESULTS

Throughout this paper, we will work in the smooth category.

Any immersion can be approximated by a self-transverse one. So we suppose that all immersions are self-transverse.

We will study multiple points and cobordism classes of orientable $4m$ -manifolds which are immersed into \mathbf{R}^{4m+2} .

Notation :

$f : M^{k(r-1)} \looparrowright \mathbf{R}^{kr}$ is an immersion of an oriented closed $k(r-1)$ -manifold in kr -space. ($r > 2$)

ν is the normal bundle of f .

e is the Euler class of ν .

w_i is the i -th Stiefel-Whitney class of M .

\overline{w}_i is the i -th normal Stiefel-Whitney class of M .

P_i is the i -th Pontryagin class of M .

\overline{P}_i is the i -th normal Pontryagin class of M .

$[M^{k(r-1)}]$ (resp. $[M^{k(r-1)}]_2$) is the fundamental homology class of $M^{k(r-1)}$ with \mathbf{Z} (resp. \mathbf{Z}_2) coefficient.

$\Theta_r(f)$ is the set of r -tuple points of f in \mathbf{R}^{kr} .

$\Delta_r(f) = f^{-1}(\Theta_r(f))$.

For f is a self transverse immersion, $\Theta_r(f)$ and $\Delta_r(f)$ are finite point sets.

If k is even, a sign can be attached to each point in $\Theta_r(f)$ by comparing the standard orientation of \mathbf{R}^{kr} with that provided by the orientation of the r normal planes at that point. We attach the same sign to each point $p \in \Delta_r(f)$ as $f(p) \in \Theta_r(f)$.

Definition 1.1. The *algebraic number of r -tuple points* of f is the number of Θ_r counted in a signed way. We write $[\Theta_r(f)]$ for it. The *algebraic number* $[\Delta_r(f)]$ is defined in the same way.

We write $[\Theta_r(f)]_2$ (resp. $[\Delta_r(f)]_2$) for the mod 2 reduction of the number of $\Theta_r(f)$ (resp. $\Delta_r(f)$).

In case k is odd, however, we cannot attach a sign to an r -tuple point, and we do not define the algebraic number of r -tuple points. In this case, the only $[\Theta_r(f)]_2$ and $[\Delta_r(f)]_2$ make sense.

In [7], Herbert proved the following;

Theorem 1.2.

$$[\Delta_r(f)]_2 = \langle \overline{w}_k^{(r-1)}, [M^{k(r-1)}]_2 \rangle_2. \quad (1.1)$$

In case k is even,

$$[\Delta_r(f)] = (-1)^{r-1} \langle e^{(r-1)}, [M^{k(r-1)}] \rangle. \quad (1.2)$$

\langle, \rangle (resp. \langle, \rangle_2) is the Kronecker product with \mathbf{Z} (resp. \mathbf{Z}_2) coefficient.

These are very simple versions of his beautiful formulae.

By definition, it is easy to see that

$$[\Delta_r(f)] = r[\Theta_r(f)]. \quad (1.3)$$

So if r is even,

$$[\Delta_r(f)]_2 = 0. \quad (1.4)$$

In [6], Felali proved the following (cf.[4]);

Theorem 1.3. *There exist an orientable $2(r-1)$ -manifold $M^{2(r-1)}$ and an immersion $f : M^{2(r-1)} \looparrowright \mathbb{R}^{2r}$ with $[\Theta_r(f)] = d$ if and only if d can be divided by $(r-1)!$.*

2. MULTIPLE POINTS OF CODIMENSION 2 IMMERSIONS

In this section, we consider the case $k = 2$ and r is odd ($r = 2m + 1$). Our aim is to prove the following theorem;

Theorem 2.1. *Let M^{4m} be a closed $4m$ -manifold and $f : M^{4m} \looparrowright \mathbb{R}^{4m+2}$ be an immersion. If M^{4m} is a spin manifold (i.e. M^{4m} is oriented and $w_2 = 0$), then the algebraic number of $(2m+1)$ -tuple points of the immersion $[\Theta_{2m+1}(f)]$ can be divided by $2^{2m}(2m)!$. Moreover, if m is odd, then $[\Theta_{2m+1}(f)]$ can be divided by $2^{2m+1}(2m)!$.*

To prove Theorem 2.1 we need two lemmas.

In [1], Atiyah and Hirzebruch proved the following lemma.

Lemma 2.2. *If M^{4m} is a spin manifold, then $\hat{A}(M^{4m})$ (the \hat{A} -genus of M^{4m}) is an integer. Moreover, if m is odd, then $\hat{A}(M^{4m})$ is an even integer.*

The total Pontryagin class of M^{4m} can be written in the form of

$$P(M^{4m}) = 1 + P_1 + P_1^2 + \cdots + P_1^m + \text{elements of order 2}, \quad (2.1)$$

because

$$T(M^{4m}) \oplus \nu = \varepsilon^{4m+2} \quad (2.2)$$

is the trivial $(4m+2)$ -bundle.

Therefore, $\hat{A}(M^{4m})$ can be represented by the P_1^m only.

The following lemma was proved in [2].

Lemma 2.3. *If M^{4m} can be immersed into \mathbf{R}^{4m+2} , then*

$$\hat{A}(M^{4m}) = \frac{(-1)^m}{2^{2m}(2m+1)!} \langle P_1^m, [M^{4m}] \rangle. \quad (2.3)$$

Now we prove our main theorem.

Proof of Theorem 2.1.

The relation between e (the Euler class of ν) and \bar{P}_1 (the first normal Pontryagin class of M^{4m}) is

$$e^2 = \bar{P}_1. \quad (2.4)$$

By Theorem 1.2 (in this case $k = 2$ and $r = 2m + 1$), we have

$$\begin{aligned} [\Delta_{2m+1}(f)] &= \langle e^{2m}, [M^{4m}] \rangle \\ &= \langle \bar{P}_1^m, [M^{4m}] \rangle \\ &= (-1)^m \langle P_1^m, [M^{4m}] \rangle. \end{aligned} \quad (2.5)$$

By (1.3)

$$[\Delta_{2m+1}(f)] = (2m+1)[\Theta_{2m+1}(f)]. \quad (2.6)$$

Thus the algebraic number of $(2m+1)$ -tuple points is

$$[\Theta_{2m+1}(f)] = \frac{(-1)^m}{2m+1} \langle P_1^m, [M^{4m}] \rangle. \quad (2.7)$$

By Lemma 2.2 and Lemma 2.3, we can easily see that $\langle P_1^m, [M^{4m}] \rangle$ can be divided by $2^{2m}(2m+1)!$. Together with (2.7), we obtain that $[\Theta_{2m+1}(f)]$ can be divided by $2^{2m}(2m)!$. In case m is odd, we can obtain the result in the same way.

This completes the proof of Theorem 2.1. \square

3. COBORDISM CLASSES OF CODIMENSION 2 IMMERSSED MANIFOLDS

Theorem 3.1. *Let $f : M^{4m} \looparrowright \mathbf{R}^{4m+2}$ and $g : N^{4m} \looparrowright \mathbf{R}^{4m+2}$ be immersions of oriented closed $4m$ -manifolds.*

Then M^{4m} and N^{4m} are oriented cobordant if and only if $[\Theta_{2m+1}(f)] = [\Theta_{2m+1}(g)]$.

In particular, M^{4m} is oriented cobordant to 0 if and only if $[\Theta_{2m+1}(f)] = 0$.

Proof of Theorem 3.1.

At first we want to show that M^{4m} and N^{4m} are unoriented cobordant to 0.

By (2.2), the total Stiefel-Whitney class of M^{4m} is

$$w(M^{4m}) = 1 + w_2 + w_2^2 + \cdots + w_2^{2m}. \quad (3.1)$$

Thus the only non-trivial Stiefel-Whitney number of M^{4m} is $\langle w_2^{2m}, [M^{4m}]_2 \rangle_2$.

For $2m + 1$ is odd,

$$[\Delta_{2m+1}(f)]_2 = [\Theta_{2m+1}(f)]_2.$$

By (1.1)

$$\begin{aligned} [\Delta_{2m+1}(f)]_2 &= \langle \bar{w}_2^{2m}, [M^{4m}]_2 \rangle_2 \\ &= \langle w_2^{2m}, [M^{4m}]_2 \rangle_2. \end{aligned}$$

Theorem 1.3 implies that $[\Theta_{2m+1}(f)]$ is even, so

$$\langle w_2^{2m}, [M^{4m}]_2 \rangle_2 = 0. \quad (3.2)$$

Therefore, M^{4m} is unoriented cobordant to 0. And so is N^{4m} .

By (2.1), the only non trivial Pontryagin number of M^{4m} (resp. N^{4m}) is $\langle P_1^m, [M^{4m}] \rangle$ (resp. $\langle P_1^m, [N^{4m}] \rangle$). Thus we can see that M^{4m} and N^{4m} are oriented cobordant if and only if $\langle P_1^m, [M^{4m}] \rangle = \langle P_1^m, [N^{4m}] \rangle$. By (2.7), the latter condition is equivalent to saying that the algebraic number of $(2m + 1)$ -tuple points of f and g attain the same value (i.e. $[\Theta_{2m+1}(f)] = [\Theta_{2m+1}(g)]$).

In particular, M^{4m} is oriented cobordant to 0 if and only if $[\Theta_{2m+1}(f)] = 0$. This completes the proof of Theorem 3.1. \square

Remark 3.2. Stong [12] proved that if M^n is an oriented closed n -manifold immersed in \mathbf{R}^{n+2} , then M^n is unoriented cobordant to 0.

Moreover, he proved that if $n \not\equiv 0 \pmod{4}$, then M^n is oriented cobordant to 0. Here we gave a proof to the first assertion for completeness.

Corollary 3.3. M^{4m} is as in Theorem 3.1.

If M^{4m} satisfies the following conditions (1) or (2), then M^{4m} is oriented cobordant to 0.

- (1) M^{4m} can be immersed in \mathbf{R}^{4m+2} with less than $(2m)!$ $(2m+1)$ -tuple points.
- (2) There exists an integer i such that $0 < i < 2m$ and $H_{2i}(M^{4m}; \mathbf{Z})$ has no free part.

Proof of Corollary 3.3.

Case (1). $[\Theta_{2m+1}(f)]$ is divided by $(2m)!$. Thus if the number of $2m+1$ -tuple points is less than $(2m)!$, then $[\Theta_{2m+1}(f)] = 0$. Therefore, M^{4m} is oriented cobordant to 0 by Theorem 3.1.

Case (2). If such an i exists, then e^{2m} is a torsion element.

Thus

$$[\Theta_{2m+1}(f)] = \frac{1}{2m+1} \langle e^{2m}, [M^{4m}] \rangle$$

must be 0. Therefore, M^{4m} is oriented cobordant to 0 by Theorem 3.1. \square

Remark 3.4. If M^{4m} is not oriented cobordant to 0, then the number of $(2m+1)$ -tuple points is more than or equal to $(2m)!$.

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REFERENCES

- [1] M. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281
- [2] ———, *Charakteristische Klassen und Anwendungen*, Enseignement Math. **7** (1961), 188–213
- [3] N. Boudriga and S. Zarati, *Points multiples isolés d'immersion de codimension 2*, C. R. Acad. Sci. Paris Ser. I Math. **296** no.14 (1983), 573–576
- [4] P.J. Eccles, *Characteristic numbers of immersions and self-intersection manifolds*, Topology with Applications, (Szegszárd 1993), Bolyai Soc. Math. Stud **4** (1995), 197–216

- [5] P.J. Eccles and W.P.R. Mitchell, *Triple points of immersed orientable $2n$ -manifolds in $3n$ -space*, J. London Math. Soc. **39**(2) (1989), 335–346
- [6] K.S. Felali, *Intersection points of immersed manifolds*, Ph. D. thesis, University of Manchester (1982)
- [7] R.J. Herbert, *Multiple points of immersed manifolds*, Memoirs AMS no.250 **34** (1981)
- [8] J. F. Hughes, *Triple points of immersed $2n$ -manifolds in $3n$ -space*, Quart. J. Math. Oxford (2) **34** (1983), 427–431
- [9] H.B. Lawson JR. and L. Michelson, *Clifford bundles, immersions of manifolds and the vector field problem*, J. Differential Geom. **15** (1980), 237–267
- [10] ———, *Spin Geometry*, Princeton Mathematical Series **38** (1989), Princeton University Press
- [11] J.W. Milnor and J. Stasheff, *Characteristic Classes*, Ann. of Math. Studies no.76 (1974) Princeton University Press
- [12] R.E. Stong, *Manifolds which immerse in small codimension*, Illinois J. Math. **27** (1983), 182–223
- [13] J.H. White, *Twist invariants and the Pontryagin numbers of immersed manifolds*, Proc. Sympos. Pure Math. **27** (1975), 429–437

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